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LETTER TO THE EDITOR

Invalidity of the replica trick for a two-dimensional fermion model

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Abstract. Non-interacting fermions in a random potential are considered on a square lattice. Such a model was previously proposed as a description of the quenched thermodynamics of a disordered Ising model. We compare in a perturbative approach the replica trick and a formal functional integral representation and find that they yield different results.

Fermion field theories are widely used to model low-dimensional systems of statistical and solid state physics near phase transition points [1]. Of particular interest are random models, since they presumably describe a number of phenomena as phase transitions in disordered spin systems or the integer quantum Hall effect [2]. Most of the investigations in this field are based on the replica trick. This method works successfully in various boson field theories. Well known examples are models for polymer chains [3] and non-interacting electrons in a random potential [4]. However, we will demonstrate in the following that this trick is invalid in certain random fermion models.

A simple description of fermions on a two-dimensional lattice $\Lambda \subset Z^2$ is given by the Hamiltonian

$$H = H_0 + V\sigma_0 \qquad H_0 = \sigma_1\Delta_1 + \sigma_2\Delta_2 \tag{1}$$

with Pauli matrices σ_j, the lattice differential operator Δ_j

$$\Delta_j f(x) = \tfrac{1}{2}[f(x+e_j) - f(x-e_j)] \tag{2}$$

$x \in \Lambda$, e_j the lattice unit vector in the j direction and a random potential $V(r)$. We assume for the latter a Gaussian distribution which is statistically independent on different lattice points:

$$\langle V_r \rangle = 0 \qquad \langle V_r^2 \rangle = 2g. \tag{3}$$

We will restrict the following investigations to the average Green function of our model:

$$G(m) = \frac{1}{|\Lambda|} \operatorname{Tr} \lim_{\varepsilon \to 0} \langle (H + m + \mathrm{i}\varepsilon\Delta\sigma_0)^{-1} \rangle_V. \tag{4}$$

The diagonal matrix $i\varepsilon D$ with $D_r \in \{-1, 1\}$, $D_r = -D_{r'}$ for $|r - r'| = 1$ is introduced to have a well defined inverse matrix [5].

A functional integral representation of (3) was discussed in the literature by several authors [6]. They considered a U(N) symmetric fermion action

$$S_N = (\psi, (H_0 + m)\psi) - g \sum_{r\in\Lambda} \left(\sum_{\mu,\mu'=1}^{2} \psi_{\mu r} \cdot \psi_{\mu r} \psi_{\mu' r} \cdot \psi_{\mu' r} \right) \tag{5}$$

with the fermion field

$$\psi^{\alpha\nu}_{\mu r} \qquad \alpha \in \{1, 2, \ldots, N\} \qquad \mu, \nu \in \{1, 2\}$$

and the scalar product

$$(\psi, \chi) = \sum_r \sum_\mu \psi_{\mu r} \cdot \chi_{\mu r} = \sum_r \sum_\mu \sum_\alpha \psi^{\alpha 1}_{\mu r} \chi^{\alpha 2}_{\mu r}$$

and defined a Green function

$$G_N(m) = \frac{1}{|\Lambda|} \int \sum_r \sum_\mu \psi^{\alpha 2}_{\mu r} \psi^{\alpha 1}_{\mu r} \exp(-S_N) \prod_{\alpha,\mu,\nu,r} \mathrm{d}\psi^{\alpha\nu}_{\mu r}. \tag{6}$$

Then it was argued that the Green function $G(m)$ can be obtained from (6) taking the limit $N \to 0$.

However, there is another functional integral representation which avoids this limiting process [7]. It is given by the UPL(1, 1) symmetric action

$$S = \mathrm{i}(\phi, (H_0 + m)D\phi) + g \sum_r \sum_{\mu,\mu'} \phi_{\mu r} \cdot \phi_{\mu r} \phi_{\mu' r} \cdot \phi_{\mu' r} \tag{7}$$

where the field ϕ has boson and fermion components

$$\phi^{\alpha\nu}_{\mu r} \phi^{\alpha'\nu'}_{\mu' r'} = (-1)^{\alpha\alpha'} \phi^{\alpha'\nu'}_{\mu' r'} \phi^{\alpha\nu}_{\mu r} \qquad \alpha, \mu, \nu \in \{1, 2\}$$

$$\phi^{22}_{\mu r} = (\phi^{21}_{\mu r})^*. \tag{8}$$

Thus the Green function is

$$G(m) = \frac{1}{|\Lambda|} \int \sum_r \sum_\mu \phi^{\alpha 2}_{\mu r} D_r \phi^{\alpha 1}_{\mu r} \exp(-S) \prod_{\alpha,\mu,\nu,r} \mathrm{d}\phi^{\alpha\nu}_{\mu r}. \tag{9}$$

The matrix D in S is important to ensure the existence of certain integrations over boson fields in the derivation of the functional integral. Moreover, the free field limit ($g = 0$) does not exist if D is absent such that we could not apply pertubation theory around $g = 0$. These problems are related to the fact that H_0 is antiHermitian whereas the potential V is Hermitian. Nevertheless, the introduction of D transforms the propagator $(H_0 + m)^{-1}$ into a Hermitian one, namely $((H_0 + m)D)^{-1}$. A consequence of this transformation is the reduction of the translational invariance to the sublattices with lattice constant $\sqrt{2}$ due to D.

Now we are in a position to verify or falsify the replica trick comparing (9) with the limit $N \to 0$ of (6) in terms of pertubation theory. Unfortunately, the pertubation expansions in powers of g do not exist for $m = 0$ due to singularities of the propagators. On the other hand, we are particularly interested in the long-range behaviour in the vicinity of the critical point $m = 0$. Therefore, we approximate the Fourier transformed propagator

$$(H_0 + m)_k = \begin{pmatrix} m & \mathrm{i}k_1 - k_2 \\ \mathrm{i}k_1 + k_2 & m \end{pmatrix} + \mathrm{o}(k_j^2) \tag{10}$$

by terms linear in k_j and introduce a cut-off:

$$k_1^2 + k_2^2 \leqslant \pi^2. \tag{11}$$

(5) is the N-component Gross–Neveu model with cut-off [8] in this approximation. We apply Wilson's renormalisation group method according to references [9, 10]:

(i) integration over the fields for $\pi/2 < |k_j| \leq \pi$,

(ii) scaling k_j by 2,

(iii) scaling ψ by $2^{-3/2}$.

There is a multiplicative mass renormalisation

$$m' = m(2 + o(g)) \tag{12}$$

which yields $m = 0$ as the critical point. Furthermore, the renormalisation of g in one-loop order is

$$g' = g - \frac{2\log 2}{\pi}(1-N)g^2 + o(g^3). \tag{13}$$

Thus the renormalisation group transformation in the replica limit $N = 0$ drives the coupling constant to the free field limit $g = 0$ [6].

The situation is different for the UPL(1, 1) symmetric action (7). To show this we perform the Fourier transformation of the propagator on the sublattice Λ_1 which is generated by $e_1 + e_2$ and $e_1 - e_2$($\sigma \in \{1, 2\}$ indicates the sublattice Λ_σ):

$$([(H_0 + m)D]_{k,\sigma\sigma'}) = \begin{pmatrix} \begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}_{11} & \begin{pmatrix} 0 & a_k \\ -a_k^* & 0 \end{pmatrix}_{12} \\ \begin{pmatrix} 0 & -a_k \\ a_k^* & 0 \end{pmatrix}_{21} & \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{22} \end{pmatrix} \tag{14}$$

with $a_k = -(1+\mathrm{i})k_1 + (1-\mathrm{i})k_2 + o(k_j^2)$.

We again neglect non-linear terms in k_j and choose the condition (11). The mass renormalisation is then of the same form as written in (12). The quartic interaction in (7) is, however, not closed under the renormalisation group transformation. Therefore, we must consider the more complicated interaction:

$$\sum_{R\in\Lambda_1} \sum_{\mu_1,\ldots,\mu_4=1}^{2} \sum_{\sigma_1,\ldots,\sigma_3=1}^{2} g\mu_1\sigma_1\mu_2\sigma_2\mu_3\sigma_3\mu_4\sigma_4 \phi_{\mu_1\sigma_1 R}\phi_{\mu_2\sigma_2 R}\phi_{\mu_3\sigma_3 R}\phi_{\mu_4\sigma_4 R} \tag{15}$$

with

$$\begin{aligned} g_{\mu_1\sigma_1\mu_2\sigma_2\mu_3\sigma_3\mu_4\sigma_4} = {} & \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}[\delta_{\mu_1\mu_3}\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4}(g_{10}\delta_{\sigma_1\sigma_3} + g_{20}\delta_{\sigma_1\hat{\sigma}_3}) \\ & + \delta_{\mu_1\hat{\mu}_3}\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4}(g_{11}\delta_{\sigma_1\sigma_3} + g_{21}\delta_{\sigma_1\hat{\sigma}_3})] \\ & + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}\delta_{\mu_1\hat{\mu}_2}\delta_{\sigma_1\hat{\sigma}_2}(t_{21}\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + t_{22}\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3}) \end{aligned} \tag{16}$$

where $\hat{\mu} \neq \mu$, $\hat{\sigma} \neq \sigma$.

The initial conditions in the present approach are

$$\begin{aligned} & g_{10} = g_{11} > 0 \\ & g_{20} = g_{21} = t_{21} = t_{22} = 0. \end{aligned} \tag{17}$$

The renormalisation group transformation in one-loop order conserves

$$g_{10} = g_{11} =: g_1$$

$$g_{20} = g_{21} =: g_2$$

$$t_{21} = t_{22} =: t$$

such that we find at the critical point $m=0(c=\log 2/\pi)$

$$\begin{aligned} g_1' &= g_1+2cg_1(2t+g_2) \\ g_2' &= g_2+2cg_1^2 \\ t' &= t+c(g_1^2-g_2^2). \end{aligned} \tag{18}$$

We note the plus sign in front of the g_1^2 term instead of the minus sign for the fermion model in (13). The difference occurs due to the different types of propagators in the former (i.e. Hermitian) and the latter (i.e. antiHermitian) case. The positive non-linear terms in (18) cause arbitrary increasing coupling constants under renormalisation in contrast to the decreasing behaviour found for the fermion replica model.

Our pertubative approach clears up a recent controversy about the evaluation of the Green function $G(m)$. It was conjectured that this quantity describes the quenched internal energy of the disordered Ising model in two dimensions [6]. While the calculations yield a singularity

$$G(m) \underset{m\sim 0}{\sim} m \log(|\log|m||) \tag{19}$$

by means of the fermion replica representation, a rigorous analysis based on a scaling inequality [11] and the investigation of a soluble model [5] lead to an analytic Green function. Now we can interpret the tendency of the renormalisation group transformation to a strong coupling behaviour in (18) as a reflection of the scaling inequality. Thus we conclude that there is a pertubation theory for $G(m)$ around $g=0$ which does not contradict the non-pertubative results. The reason for the invalidity of the replica trick rests on the symmetry breaking generated by the random potential V. To discuss this effect we suppose $\varepsilon=0$ in (4). Then we must shift the integration of V_r along the real axis by $\mathrm{i}sD_r(s\in\mathbb{R})$ to avoid the singularities of $(H+m)^{-1}$. The averaged quantity $\langle (H+m)^{-1}\rangle_V$ is not a continuous function of s at $s=0$; the symmetry $s\leftrightarrow -s$ is spontaneously broken. The regularisation $\mathrm{i}\varepsilon D$ plays then the role of an external symmetry breaking term which determines the sign of s:

$$\operatorname{sgn} s = \operatorname{sgn} \varepsilon \tag{20}$$

like a magnetic field in the Ising model which fixes the magnetisation below the critical temperature.

These symmetry breaking effects are not present in the replica trick version defined in (6), since the relevant terms are ignored in the limit $N\to 0$. Indeed, we can write

$$G(m)=\lim_{\varepsilon\to 0}\langle \det(H+m+\mathrm{i}\varepsilon\sigma_0 D)^{-N}\hat{G}_N(m, V, \varepsilon)\rangle_V \tag{21}$$

with

$$\hat{G}_N(m, V, \varepsilon):=\frac{1}{|\Lambda|}\int \sum_r\sum_\mu \psi_{\mu r}^{\alpha 2}\psi_{\mu r}^{\alpha 1}\exp[-(\psi, (H+m+\mathrm{i}\varepsilon\sigma_0 D)\psi)] \prod_{\alpha,\mu,\nu,r} \mathrm{d}\psi_{\mu r}^{\alpha\nu}.$$

The spontaneous symmetry breaking occurs only in the inverse determinant. Thus we neglect it when we set $N=0$ and remain with

$$\lim_{\varepsilon\to 0}\langle \hat{G}_N(m, V, \varepsilon)\rangle_V = G_N(m) \tag{22}$$

on the RHS of (21).

However, we observe that the replica trick is not a unique procedure. There are alternative formulations which contain the symmetry breaking. For instance, we could substitute

$$(\psi, (H_0 + m)\psi) \rightarrow (\psi, (H_0 + m)D\psi)$$

in (5). In that case we find the same pertubation theory as for the UPL(1, 1) symmetric representation.

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